Inspecting the phase pattern, we find that for the limit cycles we have $v < a\pi - 2b$ and $z > 2b$. Within the region defined by these inequalities the function $f(v, z)$ moving along the curve (9) changes its sign once only, namely at the point of intersection of (9) with the hyperbola (the line $z = v + 2b$ does not intersect (9) within this region). If the curves (9) and (12) intersected each other at more than two points, then the function $f(v, z)$ would change sign more than once on (9).

The author thanks N. N. Bautin for useful remarks.

BIBLIOGRAPHY

1. Kapranov, M. V., The pull-in range with phase automatic frequency control. Radiotekhnika, Vol. 2, №12, 1956.
2. Shakhgil'dian, V. V. and Liakhovkin, A. A., Phase automatic frequency control. M., Sviaz', 1966.
3. Kapranov, M. V., The pull-in range of automatic frequency control with a rectangular phase detector characteristic. Izv. vuzov, Radiotekhnika, №4, 1958.
4. Skriabin, N. N., A dynamic system with a discontinuous characteristic. PMM Vol. 32, №4, 1968.
5. Kaplanov, M. R. and Levin, V. A., Automatic Frequency Control. 3rd ed. M.-L., Gosenergoizdat, 1962.

Translated by L. K.

AN OPTIMAL TERMINAL CONTROL PROBLEM

PMM Vol. 33, №2, 1969, pp. 345-354

NGUYEN THAN BANG
(Hanoi)

(Received April 19, 1968)

The problem of choosing a law of time variation of controlling forces of bounded absolute value which ensure a minimal deviation measure at the end of the trajectory and a minimal control measure is investigated for linear systems with a fixed time of motion. It shows that a unique optimal trajectory and a unique control exist for this optimal terminal control problem. The possibility of using the Pontriagin maximum principle to solve this problem is demonstrated and the practical difficiulties of such an approach are pointed out. These difficulties can be overcome by means of the proposed approximate method for solving the two-point boundary value problem arising from the application of the maximum principle. A procedure for the practical realization of the above method on a computer is described.

1. Formulation of the problem. Let the motion of some system be described by the following differential equations with variable coefficients:

$$\frac{dx_\nu}{dt} = \sum_{\nu=1}^{n} a_{\nu\mu}(t)\, x_\mu + \sum_{\rho=1}^{m} b_{\nu\rho}(t)\, u_\rho(t) + f_\nu(t), \qquad x_\nu(t_0) = z_\nu^\circ \quad (\nu = 1, \ldots, n) \qquad (1.1)$$

Here x_ν are the phase coordinates of the system in question; $a_{\nu\mu}(t)$ and $b_{\nu\rho}(t)$ are the system parameters varying continuously with time; $f_\nu(t)$ are the prescribed external forces; $u_\rho(t)$ are the controlling forces of bounded absolute value whose law of variation is to be determined.

The vector function $\mathbf{u}(t) = \| u_p(t) \|$ $(m \times 1)$, which we shall call the "permissible control", is a measurable function which at each instant t, $t_0 \leqslant t \leqslant t_1$ belongs to a parallelepiped of the m-dimensional space of the variables $u_1, \ldots, u_m$

$$\mathbf{u}(t) \in \mathrm{U} = \{| u_\rho(t) | \leqslant U_\rho, \qquad t_0 \leqslant t \leqslant t_1 \quad (\rho = 1, \ldots, m) \tag{1.2}$$

The solution of differential equations (1.1) for some $\mathbf{u}(t) \in \mathrm{U}$ will be denoted by $x_\nu(t, u)$ $(\nu = 1, \ldots, n)$.

Let us pose the following problem. We are to choose from the class of permissible controls a law of variation of the vector function $\mathbf{u}^*(t)$ such that

$$J(\mathbf{u}^*, \mathbf{u}^*) = \min J(\mathbf{u}, \mathbf{u}) = \min \{R(\mathbf{u}, \mathbf{u}) + E(\mathbf{u}, \mathbf{u})\} \quad (\mathbf{u} \in \mathrm{U}) \tag{1.3}$$

Here

$$R(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \sum_{\nu=1}^{n} x_\nu^2(t_1, \mathbf{u}), \qquad E(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \int_{t_1}^{t_0} \Big[\sum_{\rho=1}^{m} c_\rho(t)\, u_\rho^2(t) \Big] dt \tag{1.4}$$

We assume that the time t_1 has been specified in advance, that the weight coefficients $c_\rho(t)$ are known nonnegative functions, and that these coefficients can vanish at only a finite number of points in the time interval $t_0 \leqslant t \leqslant t_1$.

We call the permissible control which solves the above problem the "optimal control", and the corresponding trajectory of system (1.1) the "optimal trajectory".

Several authors [1–4] have investigated the problem for the case where $c_\rho(t) \equiv 0$ $(\rho = 1, \ldots, m)$. The presence of the functional $E(\mathbf{u}, \mathbf{u})$ whose value is the control measure requires special investigation.

2. Problems of existence and uniqueness. Proceeding as in [5], we can readily show that our problem has one and only one optimal trajectory, and that fulfilment of the B-condition (whereby $m \leqslant n$ and the rank of the matrix $\mathrm{B}(t)$ is equal to m almost everywhere in the time interval $t_0 \leqslant t \leqslant t_1$) implies that the optimal control is unique.

3. Solution of the problem. Proceeding from system of differential equations (1.1), we can readily show that the expression for the functional $R(\mathbf{u}, \mathbf{u})$ can be rewritten as

$$R(\mathbf{u}, \mathbf{u}) = \int_{t_0}^{t_1} \Big[\sum_{\nu=1}^{n} \sum_{\mu=1}^{n} a_{\nu\mu}(t)\, x_\nu x_\mu + \sum_{\nu=1}^{n} \sum_{\rho=1}^{m} b_{\nu\rho}(t)\, x_\nu u_\rho + \sum_{\nu=1}^{n} f_\nu(t)\, x_\nu \Big] dt + \frac{1}{2} \sum_{\nu=1}^{n} (z_\nu^\circ)^2 \tag{3.1}$$

1°. Derivation of the maximum principle. Recalling the results of [6] and making use of expression (3.1), we can find the optimal control law from the maximum conditions for the function

$$H^* = \sum_{\nu=1}^{n} \psi_\nu(t) \Big[\sum_{\mu=1}^{n} a_{\nu\mu}(t)\, x_\mu + \sum_{\rho=1}^{m} b_{\nu\rho}(t)\, u_\rho + f_\nu(t) \Big] -$$
$$- \Big[\sum_{\nu=1}^{n} \sum_{\mu=1}^{n} a_{\nu\mu}(t)\, x_\nu x_\mu + \sum_{\nu=1}^{n} \sum_{\rho=1}^{m} b_{\nu\rho}(t)\, x_\nu u_\rho + \sum_{\nu=1}^{n} f_\nu(t)\, x_\nu + \frac{1}{2} \sum_{\rho=1}^{m} c_\rho(t)\, u_\rho^2 \Big] \tag{3.2}$$

with respect to the variables $u_1, \ldots, u_m$ in the domain U, where $\psi_\nu(t)$ is a nontrivial solution of the system of differential equations

$$\frac{d(\psi_\nu - x_\nu)}{dt} = - \sum_{\mu=1}^{n} a_{\mu\nu}(t)\,(\psi_\mu - x_\mu), \quad \psi_\nu(t_1) = 0 \quad (\nu = 1, \ldots, n) \tag{3.3}$$

Since the function H^* for fixed $\lambda_\nu = \psi_\nu - x_\nu$ and x_ν reaches its maximum together with the expression (3.4)

$$H(\lambda(t), u) = \sum_{\rho=1}^{m} S_\rho(t, \lambda) u_\rho - \frac{1}{2}\sum_{\rho=1}^{m} c_\rho(t) u_\rho^2, \quad S_\rho(t, \lambda) = \sum_{\nu=1}^{n} b_{\nu\rho}(t)\lambda_\nu, \quad (\rho = 1, \ldots, m)$$

we can see that if the controlling forces vary according to the law

$$w_\rho(t, \lambda) = \begin{cases} S_\rho(t, \lambda)/c_\rho(t) & \text{for } |S_\rho(t, \lambda)| < c_\rho(t) U_\rho \\ U_\rho \operatorname{sign} S_\rho(t, \lambda) & \text{for } |S_\rho(t, \lambda)| \geqslant c_\rho(t) U_\rho \end{cases} \quad (\rho = 1, \ldots, m) \tag{3.5}$$

where λ is a nontrivial solution of the associated system

$$\frac{d\lambda_\nu}{dt} = -\sum_{\mu=1}^{n} a_{\mu\nu}(t)\lambda_\mu, \quad \lambda_\nu(t_1) = -x_\nu(t_1, w) \quad (\nu = 1, \ldots, n) \tag{3.6}$$

then the function H^* reaches its absolute maximum with respect to the variables $u_1, \ldots, \ldots, u_m$ in the domain U.

We call a permissible control given by law (3.5) an "extremal control" and the corresponding trajectory of system (1.1) an "extremal trajectory". It is easy to show (e.g. see [6], pp. 202-206) that fulfilment of the generalized general position condition, i.e. if at any instant t, $t_0 \leqslant t \leqslant t_1$ and for any rib u of the parallelepiped U the vectors

$$B_1(t)u, \quad B_2(t)u, \ldots, B_n(t)u$$

are linearly independent in the phase-coordinate space, where the symbols $B_1(t)$, $B_2(t), \ldots \ldots, B_n(t)$ represent the matrices

$$B_1(t) = B(t), \quad B_j(t) = -A(t)B_{j-1}(t) + \frac{dB_{j-1}(t)}{dt} \quad (j = 2, \ldots, n)$$

then the functions $S_\rho(t, \lambda)$ $(\rho = 1, \ldots, m)$ for any nontrivial solution $\lambda(t) = \| \lambda_\nu(t) \| \| (n \times 1)$ of associated system (3.6) can vanish at only a finite number of points in the time interval $t_0 \leqslant t \leqslant t_1$. This means that provided the generalized general position condition is fulfilled, expressions (3.5) uniquely define the extremal control, which in this case is a continuous vector function of time.

It should be noted that the matrices $B_1(t), B_2(t), \ldots, B_n(t)$ can be determined only if the functions $b_{\nu\rho}(t)$ have $n-1$ derivatives, and the functions $a_{\nu\mu}(t)$ have $n-2$ derivatives. From now on we shall assume that the generalized general position condition is fulfilled for system (1.1).

2°. Sufficiency of the maximum principle. It is clear from the results of the previous subsection that the optimal control must be one of the extremal controls. Let us show that fulfilment of the generalized condition of positional generality implies that an extremal control is optimal, i.e. that the maximum principle in the case of our problem is not only a necessary, but also the sufficient, condition of optimality. To this end we introduce the functional

$$K(u, u) = \frac{1}{2}\sum_{\nu=1}^{n} x_\nu^2(t_1, u) + \frac{1}{2}\int_{t_0}^{t_1}\left[\sum_{\rho=1}^{m} c_\rho(t) u_\rho^2 - 2\frac{d\Phi^*(t)}{dt}\right] dt \tag{3.7}$$

in which the function $\Phi^*(t)$ is given by the expression

$$\Phi^*(t) = \sum_{\nu=1}^{n} x_\nu(t, u)[\lambda_\nu^*(t) + x_\nu(t_1, w^*)] \tag{3.8}$$

where $\lambda_\nu^*(t)$ is the solution of system (3.6) under the boundary conditions

$$\lambda_\nu^*(t_1) = -x_\nu(t_1, w)|_{w=w(t, \lambda^*)=w^*(t)} \quad (\nu = 1, \ldots, n) \tag{3.9}$$

The elements $w_\rho^*(t)$ of the vector function $w^*(t)$ are defined by law (3.5) for

$\lambda_\nu(t) = \lambda_\nu^*(t)$ $(v = 1, \ldots, n)$.

By virtue of boundary conditions (3.9),

$$\int_{t_0}^{t_1} \frac{d\Phi^*(t)}{dt}\, dt = -\sum_{\nu=1}^{n} z_\nu{}^\circ [\lambda_\nu{}^*(t_0) + x_\nu(t_1, \mathbf{w}^*)]$$

$$K(\mathbf{u}, \mathbf{u}) = J(\mathbf{u}, \mathbf{u}) = \sum_{\nu=1}^{n} z_\nu{}^\circ]\lambda_\nu{}^*(t_0) + x_\nu(t_1, \mathbf{w}^*)] \tag{3.10}$$

Thus, the functionals $K(\mathbf{u}, \mathbf{u})$ and $J(\mathbf{u}, \mathbf{u})$, which differ by a constant independent of the vector element $\mathbf{u}$, reach their minima simultaneously.

On the other hand, recalling (1.1), (3.6), (3.9), we find from (3.8) that

$$\frac{d\Phi^*(t)}{dt} = \sum_{\nu=1}^{n} \frac{dx_\nu(t, \mathbf{u})}{dt} [\lambda_\nu{}^*(t) + x_\nu(t_1, \mathbf{w}^*)] + \sum_{\nu=1}^{n} x_\nu(t, \mathbf{u}) \frac{d\lambda_\nu{}^*(t)}{dt} =$$

$$= \sum_{\rho=1}^{m} \Big[\sum_{\nu=1}^{n} b_{\nu\rho}(t)\, \lambda_\nu{}^*(t)\Big] \mu_\rho + \sum_{\nu=1}^{n} f_\nu(t)\, \lambda_\nu{}^*(t) + \sum_{\nu=1}^{n} \frac{dx_\nu(t, \mathbf{u})}{dt} x_\nu(t_1, \mathbf{w}^*) \tag{3.11}$$

Substituting (3.11) for $d\Phi^*(t)/dt$ into the right side of (3.7), we obtain the following expression:

$$K(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \sum_{\nu=1}^{n} [x_\nu(t_1, \mathbf{u}) - x_\nu(t_1, \mathbf{w}^*)]^2 -$$

$$-\int_{t_0}^{t_1} \Big\{\sum_{\rho=1}^{m} \Big[\sum_{\nu=1}^{n} b_{\nu\rho}(t)\, \lambda_\nu{}^*(t)\Big] u_\rho - \frac{1}{2} \sum_{\rho=1}^{m} c_\rho(t)\, u_\rho{}^2 \Big\}\, dt + K^* \tag{3.12}$$

where K^* denotes the following constant independent of the vector element $\mathbf{u}$:

$$K^* = \sum_{\nu=1}^{n} z_\nu{}^0 x_\nu(t_1, \mathbf{w}^*) - \frac{1}{2} \sum_{\nu=1}^{n} x_\nu{}^2(t_1, \mathbf{w}^*) - \int_{t_0}^{t_1} \Big[\sum_{\nu=1}^{n} f_\nu(t)\, \lambda_\nu{}^*(t)\Big]\, dt. \tag{3.13}$$

Expression (3.12) implies that the functional $K(\mathbf{u}, \mathbf{u})$, and therefore the functional $J(\mathbf{u}, \mathbf{u})$, reaches its minimum if and only if $u_\rho = u_\rho{}^*(t)$ for all $(\rho = 1, \ldots, m)$, i.e. if the extremal control is optimal, QED.

3°. Discussion of the results. To find the optimal control law in accordance with the above results we solve system (1.1) for $u_\rho(t) = w_\rho(t, \lambda)$ $(\rho = 1, \ldots, m)$ simultaneously with associated system (3.5); the functions $w_\rho(t, \lambda)$ are defined by law (3.5).

From the mathematical standpoint we are dealing with a nonlinear boundary value problem for which no effective method of solution has thus far been developed. The chief difficulty lies in finding the initial conditions for the ancilliary variables λ_ν ensuring fulfilment of the necessary boundary conditions in (3.6). Determination of these constants is an independent problem which, as is shown in [7], can be solved by the search method [8] on ordinary electronic analog computers. However, the presence of nonlinearities of the (3.5) type makes it difficult to prove the convergence of this method (in fact, its convergence has not yet been proved).

This obliges us to seek a different approach to the solution of our problem. Specifically, instead of finding the optimal control law directly, we shall first construct a minimizing sequence of permissible controls.

By a "minimizing sequence" we mean a sequence of permissible controls such that the corresponding sequence of values of the functional J (u, u) decreases strictly towards

$$J(\mathbf{u}^*, \mathbf{u}^*) = \min J(\mathbf{u}, \mathbf{u}) \quad (\mathbf{u} \in \mathbf{U})$$

in the limiting case.

The idea underlying this method of successive approximations was first proposed by Dem'ianov in [4 and 9] and developed in [10]. However, since the technique as worked out in these papers concerns minimization with respect to the deviation measure only, it cannot be applied directly because of the presence of the functional E (u, u) in the optimality criterion. Some further analysis is required before such an application can be made.

4. The method of successive approximations. Let us take $u_\rho(t) = u_\rho^\circ(t)$ $(\rho = 1, \ldots, m)$ as our zeroth approximation. Here $u_\rho^\circ(t)$ are arbitrary measurable functions of time which assume values from the domain U at every instant t, $t_0 \leqslant t \leqslant t_1$. As already noted, the solution of system (1.1) for $u_\rho(t) = u_\rho^\circ(t)$ will be denoted by $\mathbf{x}(t, \mathbf{u}^\circ) = \|x_\nu(t, \mathbf{u}^\circ)\|$ $(n \times 1)$. Solving the ancillary adjoint system

$$\frac{d\lambda_\nu^\circ}{dt} = -\sum_{\mu=1}^{n} a_{\mu\nu}(t)\, x_\mu, \qquad \lambda_\nu^\circ(t_1) = -x_\nu(t_1, \mathbf{u}^\circ) \qquad (\nu = 1, \ldots, n) \tag{4.1}$$

we obtain the law of variation of the function $\lambda_\nu^\circ(t)$ $(\nu = 1, \ldots, n)$.

Now let us consider the vector function $\mathbf{v}^\circ(t) = \|v_\rho^\circ(t)\|$ $(m \times 1)$, where the elements $v_\rho^\circ(t)$ are given by the formulas

$$v_\rho^\circ(t) = U_\rho \operatorname{sign}[S_\rho(t, \lambda^\circ] - c_\rho(t)\, u_\rho^\circ(t)] \quad (\rho = 1, \ldots, m) \tag{4.2}$$

and the vector function $\mathbf{w}^\circ(t) = \|w_\rho^\circ(t)\|$ $(m \times 1)$, where the $w_\rho^\circ(t)$ are defined by the law

$$w_\rho^\circ(t) = \begin{cases} S_\rho(t, \lambda^\circ)/c_\rho(t) & \text{for } |S_\rho(t, \lambda^\circ)| < c_\rho(t), (U)_\rho \\ U_\rho \operatorname{sign} S_\rho(t, \lambda^\circ) & \text{for } |S_\rho(t, \lambda^\circ)| \geqslant c_\rho(t)\, U_\rho \end{cases} \qquad (\rho = 1, \ldots, m) \tag{4.3}$$

It is not difficult to show that

$$J(\mathbf{u}^\circ, \mathbf{v}^\circ) = \min J(\mathbf{u}^\circ, \mathbf{u}) \leqslant J(\mathbf{u}^\circ, \mathbf{u}^\circ) \leqslant J(\mathbf{u}^\circ, \mathbf{u}^\circ) \quad (\mathbf{u} \in \mathbf{U}) \tag{4.4}$$

The validity of this statement follows directly from the theorem which is formulated and proved in the Appendix.

Relation (4.4) implies that two cases are possible: either $J(\mathbf{u}^\circ, \mathbf{v}^\circ) = J(\mathbf{u}^\circ, \mathbf{u}^\circ)$ or $J(\mathbf{u}^\circ, \mathbf{v}^\circ) < J(\mathbf{u}^\circ, \mathbf{u}^\circ)$. In the first of these cases it is easy to show that $\mathbf{u}^\circ(t)$ is the optimal control, and that the process is at an end. In the second case we take the following expression as our optimal control law:

$$\mathbf{u}^1(t) = \begin{cases} \mathbf{v}^\circ(t) & , \quad \text{if} \quad J(\mathbf{v}^\circ, \mathbf{v}^\circ) \leqslant J(\mathbf{u}^\circ, \mathbf{v}^\circ) \\ \alpha_0 \mathbf{u}^\circ(t) + (1 - \alpha_0)\, \mathbf{v}^\circ(t), & \quad \text{if} \quad J(\mathbf{v}^\circ, \mathbf{v}^\circ) > J(\mathbf{u}^\circ, \mathbf{v}^\circ) \end{cases} \tag{4.5}$$

where the quantity α_0 is given by the formula

$$0 < \alpha_0 = \frac{J(\mathbf{v}^\circ, \mathbf{v}^\circ) - J(\mathbf{u}^\circ, \mathbf{v}^\circ)}{J(\mathbf{v}^\circ, \mathbf{v}^\circ) - 2J(\mathbf{u}^\circ, \mathbf{v}^\circ) + J(\mathbf{u}^\circ, \mathbf{u}^\circ)} < 1 \tag{4.6}$$

It is not difficult to show that with the first approximation chosen in this way we have

$$J(\mathbf{u}^1, \mathbf{u}^1) < J(\mathbf{u}^\circ, \mathbf{u}^\circ) \tag{4.7}$$

Thus, if the zeroth approximation is not optimal, then it is always possible to choose the first approximation in such a way that condition (4.7) is fulfilled.

Let us assume that the kth approximation has already been determined, i.e. that we already know the vector functions $\mathbf{u}^k(t)$, $\mathbf{x}(t, \mathbf{u}^k) = \|x_\nu(t, \mathbf{u}^k)\|$ $(n \times 1)$, where

$x_\nu(t, \mathbf{u}^k)$ is the solution of system (1.1) for $u_\rho(t) = u_0^{\ k}(t)$ $(\rho = 1, \ldots, m)$. Solving the ancillary associated system

$$\frac{d\lambda_\nu^{\ k}}{dt} = -\sum_{\mu=1}^{n} a_{\mu\nu}(t)\,\lambda_\mu^{\ k}, \qquad \lambda_\nu^{\ k}(t_1) = -x_\nu(t_1, \mathbf{u}^k) \qquad (\nu = 1, \ldots, n) \tag{4.8}$$

we obtain the law of variation of the ancillary functions $\lambda_\nu^{\ k}(t)$ $(\nu = 1, \ldots, n)$. Now let us consider the vector function $\mathbf{v}^k(t) = \|v_\rho^{\ k}(t)\|$ $(m \times 1)$, where the elements $v_\rho^{\ k}(t)$ are given by the formulas

$$v_\rho^{\ k}(t) = U_\rho \operatorname{sign}[S_\rho(t, \boldsymbol{\lambda}^k) - c_\rho(t)\,u_\rho^{\ k}(t)] \quad (\rho = 1, \ldots, m) \tag{4.9}$$

and the vector function $\mathbf{w}^k(t) = \|w_\rho^{\ k}(t)\|$ $(m \times 1)$, where $w_\rho^{\ k}(t)$ is defined by the law

$$w_\rho^{\ k}(t) = \begin{cases} S_\rho(t, \boldsymbol{\lambda}^k)/c_\rho(t) & \text{for } |S_\rho(t, \boldsymbol{\lambda}^k)| < c_\rho(t)\,U_\rho \\ U_\rho \operatorname{sign} S_\rho(t, \boldsymbol{\lambda}^k) & \text{for } |S_\rho(t, \boldsymbol{\lambda}^k)| \geqslant c_\rho(t)\,U_\rho \end{cases} \qquad (\rho = 1, \ldots m) \tag{4.10}$$

Just as in the case of the zeroth approximation, we have

$$J(\mathbf{u}^k, \mathbf{v}^k) = \min J(\mathbf{u}^k, \mathbf{u}) \leqslant J(\mathbf{u}^k, \mathbf{w}^k) \leqslant J(\mathbf{u}^k, \mathbf{u}^k) \qquad (\mathbf{u} \in \mathbf{U}) \tag{4.11}$$

If $J(\mathbf{u}^k, \mathbf{v}^k) = J(\mathbf{u}^k, \mathbf{u}^k)$, then $\mathbf{u}^k(t)$ is the optimal control and the process is at an end. If $J(\mathbf{u}^k, \mathbf{v}^k) < J(\mathbf{u}^k, \mathbf{u}^k)$, we take the following expression as our optimal control law:

$$\mathbf{u}^{k+1}(t) = \begin{cases} \mathbf{v}^k(t), & \text{if} \quad J(\mathbf{v}^k, \mathbf{v}^k) \leqslant J(\mathbf{u}^k, \mathbf{v}^k) \\ \alpha_k \mathbf{u}^k(t) + (1 - \alpha_k)\,\mathbf{v}^k(t), & \text{if} \quad J(\mathbf{v}^k, \mathbf{v}^k) > J(\mathbf{u}^k, \mathbf{v}^k) \end{cases} \tag{4.12}$$

where the quantity α_k is given by

$$0 < \alpha_k = \frac{J(\mathbf{v}^k, \mathbf{v}^k) - J(\mathbf{u}^k, \mathbf{v}^k)}{J(\mathbf{v}^k, \mathbf{v}^k) - 2J(\mathbf{u}^k, \mathbf{v}^k) + J(\mathbf{u}^k, \mathbf{u}^k)} < 1. \tag{4.13}$$

With the $(k+1)$-th approximation chosen in this manner it is easy to show that

$$J(\mathbf{u}^{k+1}, \mathbf{u}^{k+1}) < J(\mathbf{u}^k, \mathbf{u}^k) \tag{4.14}$$

The resulting series of permissible controls $\{u^k(t)\}$ and the corresponding sequence of trajectories $\{x(t, u^k)\}$ of system (1.1) are such that

$$J(\mathbf{u}^\circ, \mathbf{u}^\circ) > J(\mathbf{u}^1, \mathbf{u}^1) > \ldots > J(\mathbf{u}^k, \mathbf{u}^k) > \ldots \tag{4.15}$$

To prove the fact that the sequences $\{\mathbf{u}^k(t)\}$ and $\{\mathbf{x}(t, \mathbf{u}^k)\}$ are, in fact, minimizing sequences in the above sense, we must show that the limit of strongly decreasing sequence (4.15) is the smallest of all possible values of the functional $J(\mathbf{u}, \mathbf{u})$. We denote the latter by J^*. The following relation is valid:

$$\lim_{k\to\infty}[\min J(\mathbf{u}^k, \mathbf{u})] = \inf[\lim_{k\to\infty} J(\mathbf{u}^k, \mathbf{v}^k)] \geqslant J^* \qquad (\mathbf{u} \in \mathbf{U}) \tag{4.16}$$

The proof of this statement follows closely the proof of the analogous statement in the problem of approximate realization of motion along a prescribed trajectory [10], and will therefore be omitted.

It is now easy to show that

$$J^* = \lim_{k\to\infty} J(\mathbf{u}^k, \mathbf{u}^k) = \min J(\mathbf{u}, \mathbf{u}) \qquad (\mathbf{u} \in \mathbf{U}) \tag{4.17}$$

Let us assume that this is not so, i.e. that there exists a permissible control $\mathbf{v}(t) \in U$ such that

$$J(\mathbf{v}, \mathbf{v}) = J^* - \varepsilon, \qquad \varepsilon > 0$$

Then, by virtue of the fact that the functional $J(\mathbf{u}^k, \mathbf{u}^k) - 2J(\mathbf{u}^k, \mathbf{v}) + J(\mathbf{v}, \mathbf{v})$ is nonnegative, we have $J(\mathbf{u}^k, \mathbf{v}) \leqslant {}^1/_2\, J(\mathbf{u}^k, \mathbf{u}^k) + J(\mathbf{v}, \mathbf{v})] = J^* + {}^1/_2\,\varepsilon_* - {}^1/_2\,\varepsilon$

Since $\varepsilon_k \to 0$ as $k \to \infty$, it follows that for sufficiently large k we have

$$\min J(\mathbf{u}^k, \mathbf{u}) \leqslant J(\mathbf{u}^k, \mathbf{v}) < J^* \qquad (\mathbf{u} \in U),$$

which contradicts inequality (4.16). This contradiction means that our assumption is invalid. Relation (4.17) has therefore been proved.

Making use of the results obtained in [6] (pp. 146, 147) in proving the existence theorems for the time-optimal operation problem and retracing the argument of [10], we can readily show that among the sequences $\{\mathbf{u}^k(t)\}$ and $\{\mathbf{x}(t, \mathbf{u}^k)\}$ there exists a pair of convergent sequences $\{\mathbf{u}^{kl}(t)\}$ and $\{\mathbf{x}(t, \mathbf{u}^{kl})\}$, whose limits have the following properties : $\mathbf{u}^*(t) \in U$, and $\mathbf{x}(t, \mathbf{u}^*)$ is a continuous vector function. Further, again following the reasoning of [10], we can readily show that the limit vector function $\mathbf{x}(t, \mathbf{u}^*)$ is unique and independent of the choice of the above pair of convergent subsequences.

Taking the limit as $l \to \infty$, we obtain

$$J^* = \min_{\mathbf{u} \in U} J(\mathbf{u}, \mathbf{u}) = \lim_{l \to \infty} J(\mathbf{u}^{k_l}, \mathbf{u}^{k_l}) = J(\mathbf{u}^*, \mathbf{u}^*) \tag{4.18}$$

Thus, the resulting sequences $\{\mathbf{u}^k(t)\}$ and $\{\mathbf{x}(t, \mathbf{u}^k)\}$ are, in fact, minimizing sequences.

Recalling relations (4.11), (4.16), (4.18) and making use of (4.10), we see that for those ρ for which the measure of the set of zeros of the function

$$S_\rho(t, \lambda^*) = \sum_{\nu=1}^{n} b_{\nu\rho}(t)\, \lambda_\nu^*(t) \qquad (\rho = 1, \ldots, m) \tag{4.19}$$

is equal to zero, the corresponding functions $w_\rho^{kl}(t)$ tend to the unique limit functions

$$u_\rho^*(t) = \begin{cases} S_\rho(t, \lambda^*) / c_\rho(t) & \text{for } |S_\rho(t, \lambda^*)| < c_\rho(t)\, U_\rho \\ U_\rho \operatorname{sign} S_\rho(t, \lambda^*) & \text{for } |S_\rho(t, \lambda^*)| \geqslant c_\rho(t)\, U_\rho \end{cases} \quad (\rho = 1, \ldots, m) \tag{4.20}$$

where $\lambda_\nu^*(t)$ is the solution of the associated system

$$\frac{d\lambda_\nu^*}{dt} = -\sum_{\mu=1}^{n} a_{\mu\nu}(t)\, \lambda_\mu^*, \qquad \lambda_\nu^*(t_1) = -x_\nu(t_1, \mathbf{u}^*) \qquad (\nu = 1, \ldots, n) \tag{4.21}$$

Thus, the limit vector function $\mathbf{u}^*(t)$, which is the optimal control by virtue of relation (4.18), satisfies the Pontriagin maximum principle.

The statements proved above do not directly imply the uniqueness of the optimal control. However, as was pointed out in Sect. 2, fulfilment of the B-condition implies the uniqueness of the optimal control by virtue of the uniqueness of the optimal trajectory. The greatest difficulties in approximate calculation of the optimal control law by this method have to do with the solution of the system of the (4.8) type. We know only the values of the variables λ_ν^k at the instant $t = t_1$, so that determination of the law of variation of the functions $\lambda_\nu^k(t)$ on a computer requires that we begin by determining the corresponding initial conditions for these variables. As is shown in [11–13], the latter can be found by integrating system (4.8) by "working backwards", i.e. by introducing a new independent variable by means of the relation $t = t_1 + t_0 - \sigma$. This transforms system (4.8) into

$$\frac{d\Lambda_\nu^k(\sigma)}{d\sigma} = \sum_{\mu=1}^{n} a_{\mu\nu}(t_1 + t_0 - \sigma)\, \Lambda_\mu^k(\sigma), \qquad \Lambda_\nu^k(t_0) = -x_\nu(t_1, \mathbf{u}^k) \quad (\nu = 1, \ldots, n) \tag{4.22}$$

Since $\Lambda_\nu^k(\sigma) = \lambda_\nu^k(t_1 + t_0 - \sigma)$, it is obvious that the values of the function $\Lambda_\nu^k(\sigma)$ at the instant $\sigma = t_1$ serve as the initial conditions for the variables λ_ν^k which ensure the fulfilment of the boundary conditions in (4.8).

This enables us to propose the following computation procedure.

1°. We choose some permissible control $\mathbf{u}^k(t) = \|u_\rho{}^k(t)\|$ $(n \times 1) \in U$ and solve system (1.1) for $u_\rho(t) = u_\rho{}^k(t)$. The quantities $x_\nu(t, \mathbf{u}^k)$ and $J(\mathbf{u}^k, \mathbf{u}^k)$ are found in the course of integration.

2°. We integrate system (4.22) in the limiting case for $t_0 \leqslant \sigma \leqslant t_1$ and memorize the values of the variables $\Lambda_\nu{}^k$ at the instant $\sigma = t_1$.

3°. We solve system (4.8) under the initial conditions $\Lambda_\nu{}^k(t_0) = \Lambda_\nu{}^k(t_1)$ and at the same time determine the elements $v_\rho{}^k(t)$ of the vector function $v^k(t) = \|v_\rho{}^k(t)\|$ $(m \times 1)$ by means of formula (4.9).

4°. We integrate system (1.1) for $u_\rho(t) = v_\rho{}^k(t)$ and at the same time compute the quantities $J(\mathbf{u}^k, \mathbf{v}^k)$ and $J(\mathbf{v}^k, \mathbf{v}^k)$.

5°. We compare the quantity $J(\mathbf{u}^k, \mathbf{v}^k)$ with the quantity $J(\mathbf{u}^k, \mathbf{u}^k)$; the computation process ends as soon as $J(\mathbf{u}^k, \mathbf{u}^k) = J(\mathbf{u}^k, \mathbf{v}^k)$. If this does not occur, we compute the next approximation from formulas (4.12) and (4.13) by means of ancillary functional blocks, and the process begins again.

Appendix. To prove the theorem used in constructing the minimizing sequence of permissible controls we must first establish the validity of the following lemmas.

Lemma 1. Let $\mathbf{u}^j(t)$ be any prescribed measurable vector function which assumes values in the domain U at every instant t, $t_0 \leqslant t \leqslant t_1$; let $v^j(t) = \|v_\rho{}^j(t)\|(m \times 1)$ be the permissible control defined by the law

$$v_\rho{}^j(t) = U_\rho \operatorname{sign}[S_\rho(t, \boldsymbol{\lambda}^j) - c_\rho(t)\, u_\rho{}^j(t)], \qquad S_\rho(t, \boldsymbol{\lambda}^j) = \sum_{\nu=1}^{n} b_{\nu\rho}(t)\, \lambda_\nu{}^j(t) \qquad (\rho = 1, \ldots, n) \tag{A.1}$$

where $\lambda_\nu{}^j(t)$ is the solution of the associated system

$$\frac{d\lambda_\nu{}^j}{dt} = -\sum_{\mu=1}^{n} a_{\mu\nu}(t)\, \lambda_\mu{}^j, \qquad \lambda_\nu{}^j(t_1) = -x_\nu(t_1, \mathbf{u}^j) \quad (\nu = 1, \ldots, n) \tag{A.2}$$

This means that

$$J(\mathbf{u}^j, \mathbf{v}^j) = \min J(\mathbf{u}^j, \mathbf{u}) = \min[\mathrm{R}(\mathbf{u}^j, \mathbf{u}) + E(\mathbf{u}^j, \mathbf{u})] \qquad (\mathbf{u} \in \mathrm{U}) \tag{A.3}$$

where $R(\mathbf{u}^j, \mathbf{u})$ and $E(\mathbf{u}^j, \mathbf{u})$ denote the following functionals:

$$R(\mathbf{u}^j, \mathbf{u}) = \frac{1}{2} \sum_{\nu=1}^{n} x_\nu(t_1, \mathbf{u}^j)\, x_\nu(t_1, \mathbf{u}) = \frac{1}{2} (\mathbf{x}(t_1, \mathbf{u}^j) \cdot \mathbf{x}(t_1 \mathbf{u})) \tag{A.4}$$

$$E(\mathbf{u}^j, \mathbf{u}) = \frac{1}{2} \int_{t_0}^{t_1} \left[\sum_{\rho=1}^{m} c_\rho(t)\, u_\rho{}^j(t)\, u_\rho\right] dt \tag{A.5}$$

Proof. We begin to prove this lemma by transforming the functional $R(\mathbf{u}^j, \mathbf{u})$. As we know, the solution of system (1.1) for some $\mathbf{u}(t) \in U$ can be expressed in matrix form,

$$\mathbf{x}(t, \mathbf{u}) = \mathbf{s}(t) + \int_{t_0}^{t} \mathbf{X}(t)\, \mathbf{X}^{-1}(\sigma)\, \mathbf{B}(\sigma)\, \mathbf{u}(\sigma)\, d\sigma \tag{A.6}$$

where $\mathbf{s}(t)$ is the vector function

$$\mathbf{s}(t) = \mathbf{X}(t)\, \mathbf{z}^\circ + \int_{t_0}^{t} \mathbf{X}(t)\, \mathbf{X}^{-1}(\sigma)\, \mathbf{f}(\sigma)\, d\sigma \tag{A.7}$$

The matrix function $X(t)$ in the above expression is the normed fundamental matrix of system (1.1) for $u_\rho(t) \equiv 0$ $(\rho = 1, \ldots, m)$, $f_\nu(t) \equiv 0$ $(\nu = 1, \ldots, n)$; $\mathbf{X}^{-1}(\sigma)$ denotes the inverse of the matrix $\mathbf{X}(\sigma)$.

Substituting expression (A.6) for $t = t_1$ into the right side of expression (A.4), we obtain

$$R(\mathbf{u}^j, \mathbf{u}) = R^j - \frac{1}{2}\int_{t_0}^{t_1} (\mathbf{B}^\tau(t)\lambda^j(t)\cdot\mathbf{u}(t))\,dt \tag{A.8}$$

where R^j denotes the quantity

$$R^j = {}^1\!/_2\,(\mathbf{x}(t_1, \mathbf{u}^j)\,\mathbf{s}(t_1)) \tag{A.9}$$

independent of u, and $\lambda^j(t)$ is the vector function

$$\lambda^j(t) = [\mathbf{X}^{-1}(t)]^\tau \mathbf{X}^\tau(t_1)\,\mathbf{x}(t_1, \mathbf{u}^j) \tag{A.10}$$

It is easy to show that the vector function $\lambda^j(t)$ defined by formula (A.10) is the solution of adjoint system (A.2).

The expanded form of expression (A.8) is

$$R(\mathbf{u}^j, \mathbf{u}) = R^j - \frac{1}{2}\int_{t_0}^{t_1}\left\{\sum_{\rho=1}^{m}\left[\sum_{\nu=1}^{n} b_{\nu\rho}(t)\,\lambda_\nu^j(t)\right]u_\rho\right\}dt \tag{A.11}$$

By virtue of (A.5) and (A.11), the expression for the functional $J(\mathbf{u}^j, \mathbf{u})$ becomes

$$J(\mathbf{u}^j, \mathbf{u}) = R^j - \frac{1}{2}\int_{t_0}^{t_1}\left\{\sum_{\rho=1}^{m}\left[\sum_{\nu=1}^{n} b_{\nu\rho}(t)\,\lambda_\nu^j(t) - c_\rho(t)\,u_\rho^j(t)\right]u_\rho\right\}dt \tag{A.12}$$

The above expression shows that the functional $J(\mathbf{u}^j, \mathbf{u})$ assumes its minimum value if and only if the u_ρ vary according to law (A.1). Lemma 1 has been proved.

L e m m a 2. Let $\mathbf{u}^j(t)$ be any given measurable vector function which assumes values in the domain $\mathbf{U}$ at every instant t, $t_0 \leqslant t \leqslant t_1$ and let $\mathbf{w}^j(t) = \|\, w_\rho^j(t)\,\|$ $(m \times 1)$ be the permissible control defined by the law

$$w_\rho^j(t) = \begin{cases} S_\rho(t, \lambda^j)/c_\rho(t) & \text{for } |S_\rho(t, \lambda^j)| < c_\rho(t)\,U_\rho \\ U_\rho \operatorname{sign} S_0(t, \lambda^j) & \text{for } |S_\rho(t, \lambda^j)| \geqslant c_\rho(t)\,U_\rho \end{cases} \qquad (\rho = 1, \ldots, n) \tag{A.13}$$

where $\lambda_\nu^j(t)$ is the solution of adjoint system (A.2).

The following relation is then valid:

$$2R(\mathbf{u}^j, \mathbf{w}^j) + E(\mathbf{w}^j, \mathbf{w}^j) = \min\,[2R(\mathbf{u}^j, \mathbf{u}) + E(\mathbf{u}, \mathbf{u})] \quad (\mathbf{u} \in \mathbf{U}) \tag{A.14}$$

P r o o f. Recalling expression (A.11) for the functional $R(\mathbf{u}^j, \mathbf{u})$ we obtain the following expression for the functional $2R(\mathbf{u}^j, \mathbf{u}) + E(\mathbf{u}, \mathbf{u})$:

$$2R(\mathbf{u}^j, \mathbf{u}) + E(\mathbf{u}, \mathbf{u}) = 2R^j - \int_{t_0}^{t_1}\left\{\sum_{\rho=1}^{m}\left[\sum_{\nu=1}^{m} b_{\nu\rho}(t)\,\lambda_\nu^j(t)\right]u_\rho\right\}dt + \frac{1}{2}\int_{0}^{t_1}\left[\sum_{\rho=1}^{n} c_\rho(t)\,u_\rho^2\right]dt =$$

$$= 2R^j + \frac{1}{2}\int_{0}^{t_1}\left\{\sum_{\rho=1}^{m} c_\rho(t)\left[\left(u_\rho - \frac{1}{c_\rho(t)}\sum_{\nu=1}^{m} b_{\nu\rho}(t)\,\lambda_\nu^j(t)\right)^2 - \left(\frac{1}{c_\rho(t)}\sum_{\nu=1}^{n} b_{\nu\rho}(t)\,\lambda_\nu^j(t)\right)^2\right]\right\}dt \tag{A.15}$$

We infer from this expression that the functional $2R(\mathbf{u}^j, \mathbf{u}) + E(\mathbf{u}, \mathbf{u})$ becomes minimal if and only if $u_\rho \equiv w_\rho^j(t)$, where the functions $w_\rho^j(t)$ are given by (A.13), QED.

L e m m a 3. The following inequality is always valid for any two permissible controls $\mathbf{u}^j(t) \in \mathbf{U}$, $\mathbf{w}^j(t) \in U$:

$$2E(\mathbf{u}^j, \mathbf{w}^j) \leqslant E(\mathbf{u}^j, \mathbf{u}^j) + E(\mathbf{w}^j, \mathbf{w}^j) \tag{A.16}$$

The proof of this statement follows directly from the condition of nonnegativeness of the functional $E^j = E(\mathbf{u}^j, \mathbf{u}^j) - 2E(\mathbf{u}^j, \mathbf{w}^j) + E(\mathbf{w}^j, \mathbf{w}^j)$.

L e m m a 4. Let all the conditions of Lemma 2 be fulfilled. The following inequality is then valid:

$$J(\mathbf{u}^j, \mathbf{w}^j) \leqslant J(\mathbf{u}^j, \mathbf{u}^j) \tag{A.17}$$

Proof. Expression (A.14) implies that

$$2R(\mathbf{u}^j, \mathbf{w}^j) + E(\mathbf{w}^j, \mathbf{w}^j) \leqslant 2R(\mathbf{u}^j, \mathbf{u}^j) + E(\mathbf{u}^j, \mathbf{u}^j) \tag{A.18}$$

and by virtue of Lemma 3 we have

$$2E(\mathbf{u}^j, \mathbf{w}^j) \leqslant E(\mathbf{u}^j, \mathbf{u}^j) + E(\mathbf{w}^j, \mathbf{w}^j) \tag{A.19}$$

Adding (A.18) and (A.19), we obtain

$$2R(\mathbf{u}^j, \mathbf{w}^j) + 2E(\mathbf{u}^j, \mathbf{w}^j) \leqslant 2R(\mathbf{u}^j, \mathbf{u}^j) + 2E(\mathbf{u}^j, \mathbf{u}^j)$$

This implies the validity of Lemma 4.

The above lemmas clearly imply the following theorem.

Theorem. The following relation is valid for any given measurable vector function $\mathbf{u}^j(t)$ which assumes values in the domain U at every instant t, $t_0 \leqslant t \leqslant t_1$:

$$J(\mathbf{u}^j, \mathbf{v}^j) = \min J(\mathbf{u}^j, \mathbf{u}) \leqslant J(\mathbf{u}^j, \mathbf{w}^j) \leqslant J(\mathbf{u}^j, \mathbf{u}^j) \quad (\mathbf{u} \in \mathbf{U}) \tag{A.20}$$

where the elements of the vector functions

$$\mathbf{v}^j(t) = \| v_\rho^{\,j}(t) \| (m \times 1), \qquad \mathbf{w}^j(t) = \| w_\rho^{\,j}(t) \| (m \times 1)$$

are given by Formulas (A.1) and (A.13), respectively.

BIBLIOGRAPHY

1. Bellman, R. E., Adaptive Control. Princeton University Press, 1962.
2. Ho, Yu-chi, The method of successive approximations for optimal control systems with controlling-signal limiting. Tr. A.S.M.E., Vol. 84, №1, 1962.
3. Kirillova, L.S., The problem of optimizing the final state of a controlled system. Avtomatika i Telemekhanika Vol.23, №12, 1962.
4. Dem'ianov, V.F., Constructing an optimal program in a linear system. Avtomatika i Telemekhanika Vol.25, №1, 1964.
5. Nguyen Than Bang, Some optimal problems concerning the approximate realization of motion along a prescribed trajectory. II. Avtomatika i Telemekhanika №7, 1967.
6. Pontriagin, L.S., Boltianskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F., Mathematical Theory of Optimal Processes. Fizmatgiz, Moscow, 1961.
7. Nguyen Than Bang, The optimal problem of a "hit" in transient linear systems. Tekhnicheskaia Kibernetika, №1, 1965.
8. Fel'dbaum, A.A., Computing Devices in Automatic Systems. Fizmatgiz, Moscow, 1959.
9. Dem'ianov, V.F., The construction of an integral optimal programed control function in a linear system. PMM Vol.27, №3, 1963.
10. Nguyen Than Bang, Some optimal problems concerning the approximate realization of motion along a prescribed trajectory. I. Avtomatika i Telemekhanika №7, 1967.
11. Tsien, H.S., Engineering Cybernetics. McGraw-Hill, New York, 1954.
12. Savinov, G.V., Using analog computers to solve the problem of disturbance buildup. Vestnik MGU, Matematika i Mekhanika №3, 1961.
13. Nguyen Than Bang, Solving certain problems of dynamic programing theory by means of analog computers. Avtomatika i Telemekhanika Vol.23, №9, 1962.

Translated by A.Y.